The Isomorphism Problem for *k*-Trees is Complete for Logspace

Johannes Köbler   Sebastian Kuhnert

Nový Smokovec, August 24th, 2009

Outline

Graph isomorphisms

Definition

Let G, H be graphs

- A bijection $\varphi : V_G \to V_H$ is an *isomorphism*, if $\forall u, v \in V_G : \{u, v\} \in E(G) \Leftrightarrow \{\varphi(u), \varphi(v)\} \in E(H)$
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- If such a φ exists, G and H are *isomorphic* ($G \cong H$).
- For a graph G (w. l. o. g. $V_G = \{1, \ldots, n\}$) and $\varphi \in S_n$ let $\varphi(G)$ be the graph given by

$$V_{\varphi(G)} := \{\varphi(v) \mid v \in V_G\}$$
$$E_{\varphi(G)} := \Big\{\{\varphi(u), \varphi(v)\} \,\Big|\, \{u, v\} \in E_G\Big\}$$

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$$c_{\varphi(G)}(v) := c_G(\varphi^{-1}(v))$$

Invariants and canonical labelings

Definition

Let $\mathcal{G}$ be a graph class and f a function defined on $\mathcal{G}$.

- f is an *invariant* for $\mathcal{G}$, if

$$\forall G, H \in \mathcal{G} : G \cong H \Rightarrow f(G) = f(H) .$$

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Assume w.l.o.g. that $V_G = \{1, \ldots, n\}$.

- A function $\psi : \mathcal{G} \to S_n$ that maps $G \mapsto \psi_G$
is a *canonical labeling* for $\mathcal{G}$,
if $G \mapsto \psi_G(G)$ is a canonization for $\mathcal{G}$.

k-trees

Definition

The class of *trees* can be defined inductively:

- A single vertex is a tree.
- If G is a tree, the following construction yields a tree G':
  - choose a vertex u in G and
  - connect u with a new vertex v:

    $V_{G'} := V_G \cup \{v\}$

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- A 1-Clique is a tree.
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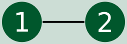
Partial k-trees are subgraphs of k-trees.

Example

- Partial 1-trees are forests.
- G is a partial k-tree iff its tree width is $\leq k$.

A 2-tree

Example



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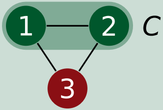

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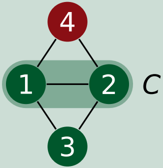

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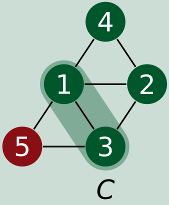

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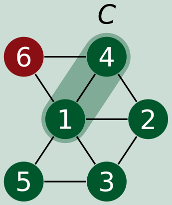

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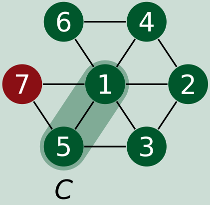

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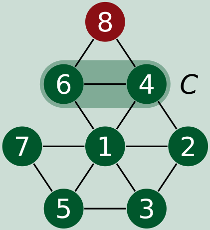

Known Results

Tree isomorphism and canonization

- In time $\mathcal{O}(n)$ [Aho, Hopcroft, Ullman 74]
- In NC [Miller, Reif 91]
- In L [Lindell 92]
- Complete for L [Jenner et al. 03]

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Partial k-tree isomorphism

- In time $O(n^{k+4.5})$ [Bodlaender 90]
- For $k = 2$ and 3 in time $\mathcal{O}(n \log n)$ [Arnborg, Proskurowski 92]
- In TC^1 [Grohe, Verbitsky 06]
- Canonizing in TC^2 [Köbler, Verbitsky 08]
- For $k = 2$ complete for L [Arvind, Das, Köbler 08]

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Outline

1. Introduction
   - Isomorphisms and Canonization
   - k-trees
   - Known Results

2. Canonizing k-trees
   - Tree representation
   - The FL algorithm

Outline of our algorithm

k-tree G

$\downarrow$

canon of G

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$$\begin{tikzcd}
k\text{-tree } G \arrow[r] & \text{tree representation } T(G) \arrow[d, "\text{variant of Lindell's algorithm}"] \\
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Requirements for this approach:

- Isomorphic k-trees must have isomorphic tree representations
- $T(G)$ must contain enough information to reconstruct an isomorphic copy of G
- Both construction and reconstruction must be possible in logspace

Tree representation of k-trees

Definition

Let G be a k-tree.
The *tree representation* $T(G)$ is defined by

$$V_{T(G)} := \left\{ M \subseteq V_G \,\middle|\, M \text{ is a } k\text{- or } (k+1)\text{-clique in } G \right\}$$

$$E_{T(G)} := \left\{ \{M_1, M_2\} \subseteq V_{T(G)} \,\middle|\, M_1 \subsetneq M_2 \right\}$$

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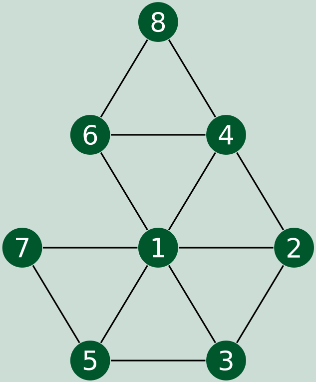

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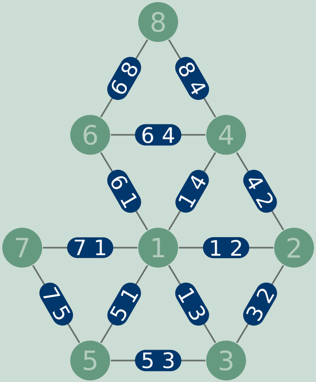

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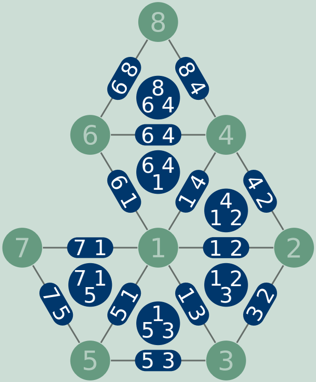

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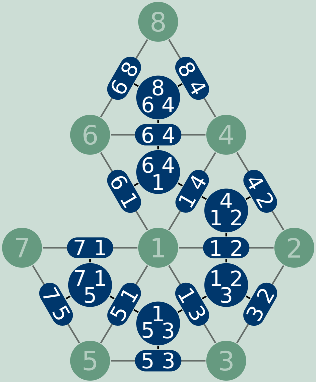

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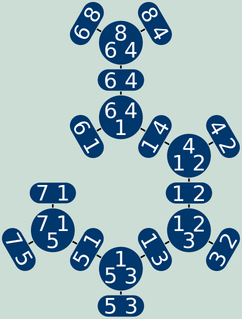

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$T(G)$:

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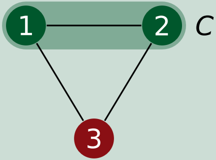

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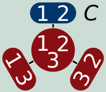

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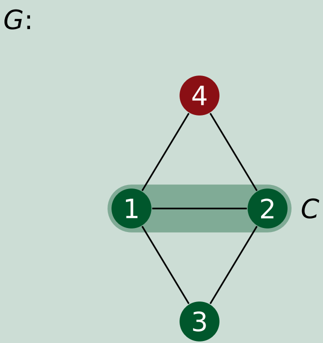

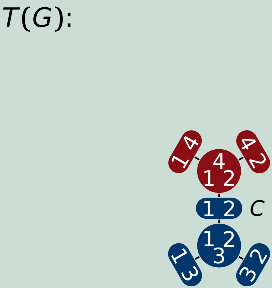

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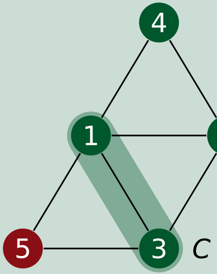

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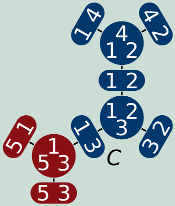

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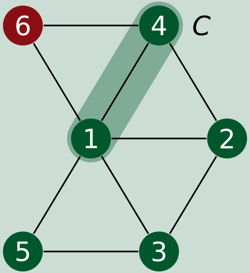

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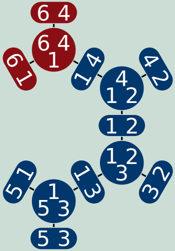

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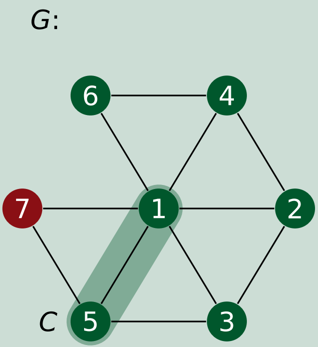

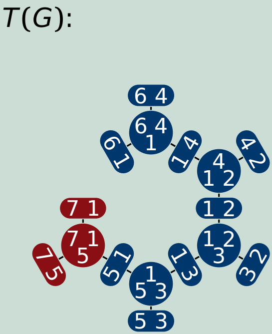

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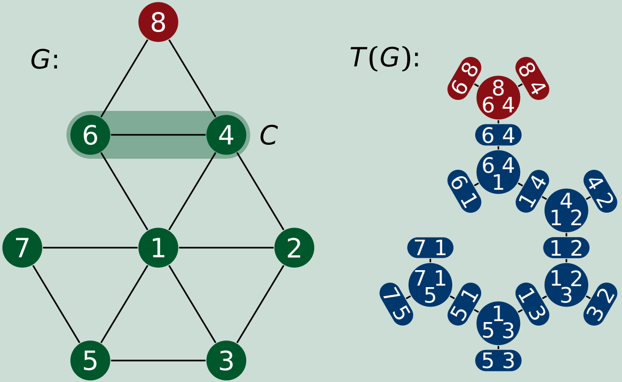

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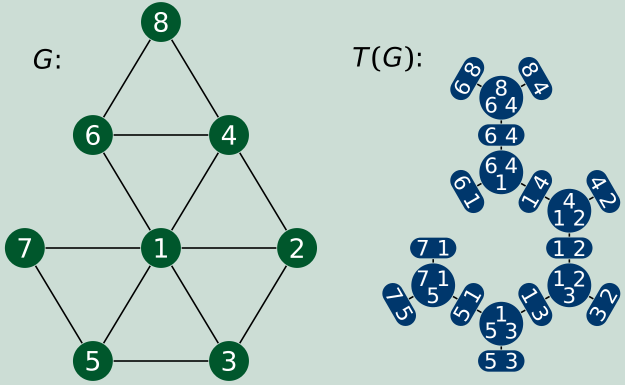

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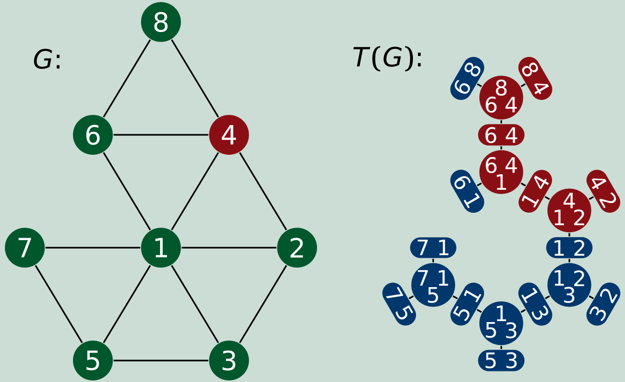
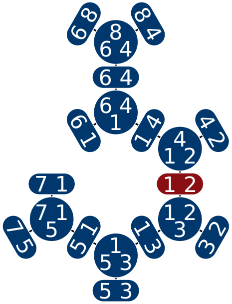


- $T(G)$ is a tree.
- For any $v \in V_G$, the nodes of $T(G)$ that contain v form a subtree of $T(G)$.

The kernel of a k-tree

Fact

For any k-tree G, the *center* *of $T(G)$ is a single node.*



Johannes Köbler, Sebastian Kuhnert

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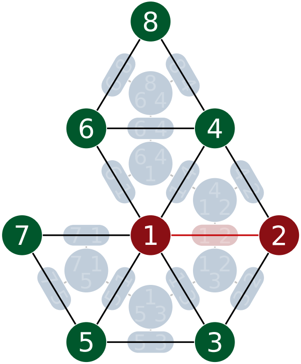

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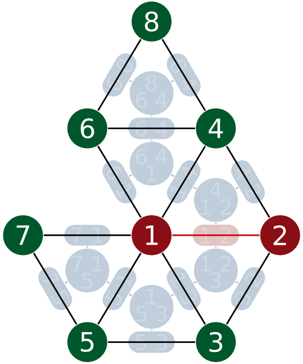

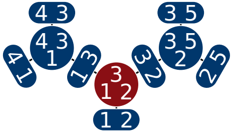

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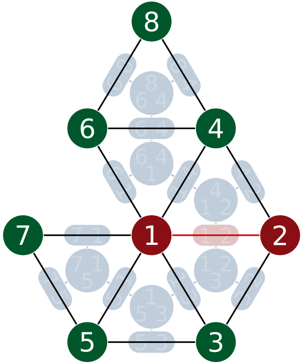
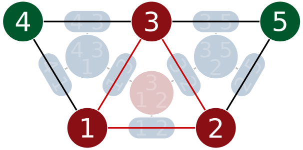
- We define $k' := \|\ker(G)\|$.





The need for colors

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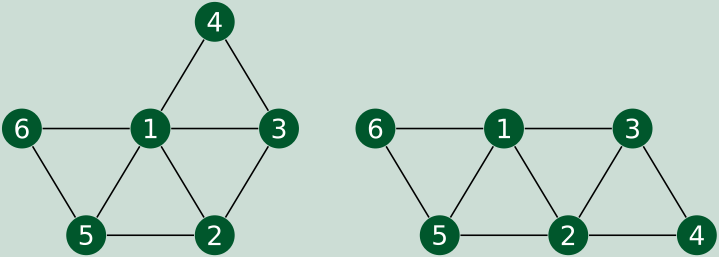

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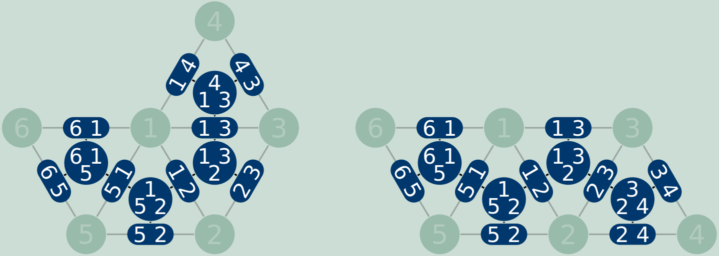

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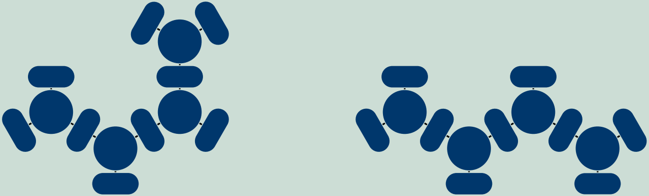

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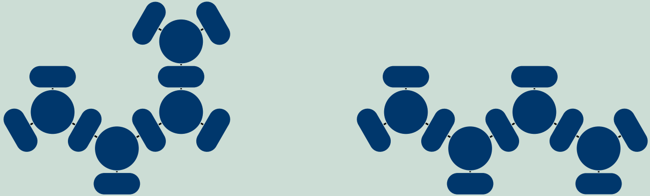

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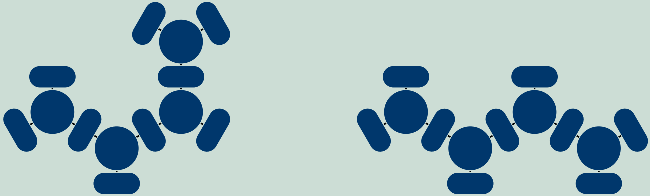

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- Thus it is impossible to reconstruct an isomorphic copy of G from an isomorphic copy of $T(G)$
- Solution: Color the nodes of $T(G)$
  to fully encode the structure of G

Coloring the tree representation

Definition

Let G be a k-tree with $V_G = \{1, \ldots, n\}$ and $K := \ker(G) = \{1, \ldots, k'\}$ and let $v \in V_G$.

- The *level* of v is

$$l(v) := \min \left\{ d_{T(G)}(K, M) \,\middle|\, M \in V_{T(G)}, v \in M \right\}$$

Now let $\pi \in S_{k'}$ be a permutation on K.

- The *color* of v is

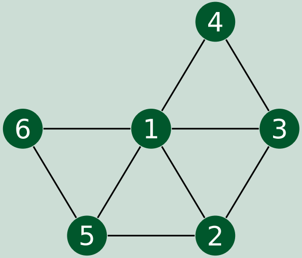
$$c_\pi(v) := \begin{cases} \pi(v) & \text{if } v \in \ker(G) \\ l(v) + k' & \text{otherwise} \end{cases}$$

- The *colored tree representation $T(G, \pi)$* of G is $T(G)$ with K as root and each $M \in V_{T(G)}$ colored by

$$c_\pi(M) := \left\{ c_\pi(v) \,\middle|\, v \in M \right\}$$

Colored tree representations

Example



Colored tree representations

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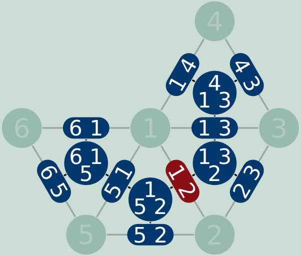

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- $l(1) = l(2) = 0$
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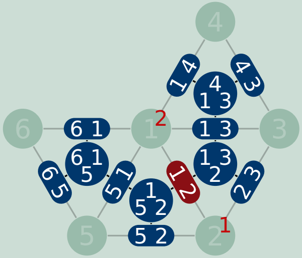

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- For $v \in \ker(G)$:
  $c_\pi(v) = \pi(v)$
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Colored tree representations

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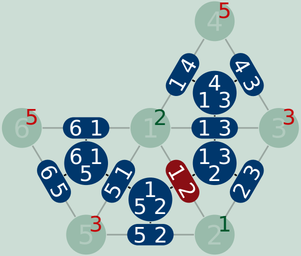

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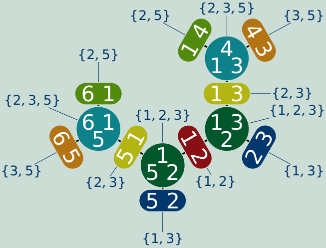

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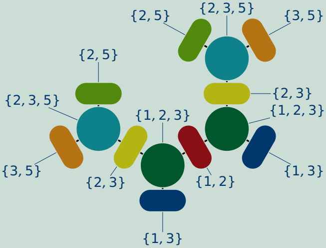

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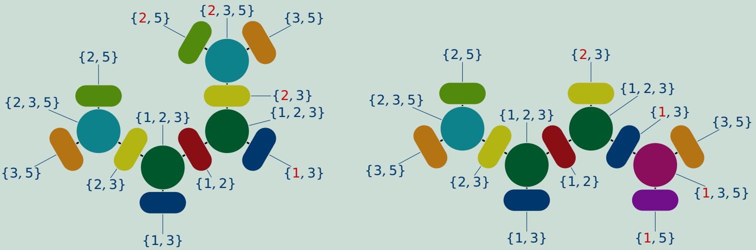
Colored tree representations

Example



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Facts on $T(G, \pi)$

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For a k-tree G and a permutation π on $\ker(G)$, $T(G, \pi)$ can be computed in FL.

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Let G, H be k-trees such that $G \cong H$ via φ, $V(G) = V(H) = \{1, \ldots, n\}$ and $\ker(G) = \ker(H) = K$. Then also $T(G, \pi_1) \cong T(H, \pi_2)$ via φ, provided that $\pi_1(u) = \pi_2(\varphi(u))$ for all $u \in K$.

Facts on $T(G, \pi)$

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Lemma

Let G be a k-tree and let π be a permutation on the kernel K of G.

- *From any colored tree $T \cong T(G, \pi)$, an isomorphic copy G' of G can be computed in FL.*
- *It is possible to compute an isomorphism between G and G' from any given isomorphism between $T(G, \pi)$ and T in FL.*

Canonical labeling for k-trees

Theorem

Given a k-tree G with $V_G = \{1, \ldots, n\}$ and kernel $K = \{1, \ldots, k'\}$, a canonical labeling $\psi_G \in S_n$ can be computed in FL.

The algorithm

```
foreach π ∈ S_k′ do
  compute T(G, π)
  compute a canonical labeling φ_T(G,π) of
      T(G, π)
choose π_1 ∈ S_k′ such that φ_T(G,π_1)(T(G, π_1)) is
    minimal
reconstruct a labeling ψ_G of G from φ_T(G,π)
return ψ_G as canonical labeling
```

Our results

Corollary

For any fixed k,

- *k-tree canonization is in FL and*
- *k-tree isomorphism is in L.*

Corollary

For any fixed k,

- *computing a generating set of $\mathrm{Aut}(G)$ for a given k-tree G is in FL,*
- *computing a canonical labeling coset for a given k-tree is in FL, and*
- *k-tree automorphism is L-complete.*

Summary and open problems

- Canonical labeling for k-trees can be reduced in logspace to canonical labeling for trees
- k-tree isomorphism is L-complete
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- What about *partial* k-tree isomorphism?
  - Can the upper bound of TC^1 be improved? (e. g. to NL, $\oplus\mathrm{L}$, L)
  - Is it hard for NL or $\oplus\mathrm{L}$?
- Can our approach be generalized?
  - To hookup classes that are not isomorphism complete, c. f. [Klawe, Corneil, Proskurowski 82]
  - To chordal graphs with small s-components, c. f. [Toda 06]

Thank You!

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